Probability Review

SW Chapter 2.1-2.4

ECON3500: Econometrics and Applications

Learning objectives

- Understand and use key vocabulary
- Calculate expected values and variances and apply their properties

Probability Review (Chapter 2.1-2.4)

- Random variables
  - Discrete distributions
  - Continuous distribution functions
- Features of probability distributions
- Joint probability distributions
- Normal distribution
  - Finding normal probabilities

Random variables

Key definitions: random variables

- Random variable: discrete and continuous
- Probability density function
- Cumulative density function
- Joint distribution

Random variables

Random variable **Definition**
Represents a possible numerical value from a random experiment:

- Discrete random variable: Takes on no more than a countable number of values.
- Continuous random variable: Can take on any value in an interval - possible values measured on a continuum.

Discrete vs. continuous random variables

Discrete

- Roll a die twice, X is number of times 4 comes up ($X \in 0, 1, 2$).
- Toss a coin five times, X is the number of heads ($X \in 0, 1, 2, 3, 4, 5$).

Continuous

- Weight of packages filled by mechanical process
- Temperature of cleaning solution
- Time between failures of an electrical component

Probability density function

Let X be a discrete random variable and x be one of the possible values.

- The probability that X takes value x is written as $P(X = x) = P(x)$.

Probability density function **Definition**
Representation of the probabilities for all possible outcomes.

- $0 \leq P(x) \leq 1$ for any value of x
- $\sum_x P(x) = 1$

Note that in the discrete case, sometimes called probability <u>distribution</u> function

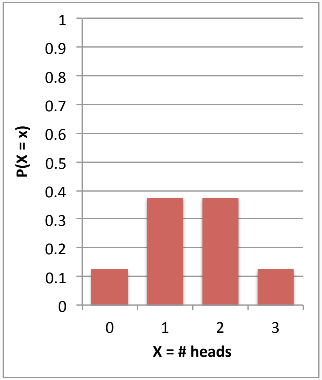
Probability distribution function: example

Example 1
Consider the following random experiment:

- Toss 3 coins.
- Define X as the number of heads.
- What is the probability distribution function of X? That is, show $P(x)$ for all values of x.

Probability density function: example

x	$P(x)$
0	$P(0) = 1/8 = 0.125$
1	$P(1) = 3/8 = 0.375$
2	$P(2) = 3/8 = 0.375$
3	$P(3) = 1/8 = 0.125$



Continuous random variables

- A continuous random variable has an **uncountable** number of values.
- Because there are infinite possible values, the probability of each individual value is infinitesimally small.
- If X is a continuous random variable, then $P(X = x) = 0$ for any individual value x.
- Only meaningful to talk about ranges.

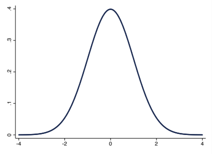
Probability density functions (PDF)

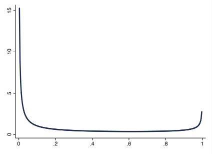
- Let X be a continuous random variable
- Its probability density function (PDF), $f(x)$ is a function that lets us compute the probability that X falls within some range of potential values.
- We define $f(x)$ such that the probability that X falls within any interval of values is equal to the *area under the curve* of $f(x)$ over that interval.

Probability density function properties

Properties of the probability density function (PDF), $f(x)$, of random variable X:

1. $f(x) > 0$ for all values of x.



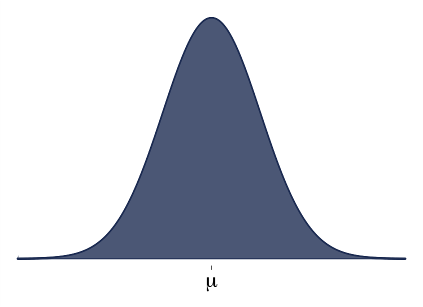


Probability density function properties

Properties of the probability density function (PDF), $f(x)$, of random variable X:

2. The area under $f(x)$ over all values of the random variable X within its range equals 1.

$$\int_X f(x)dx = 1$$

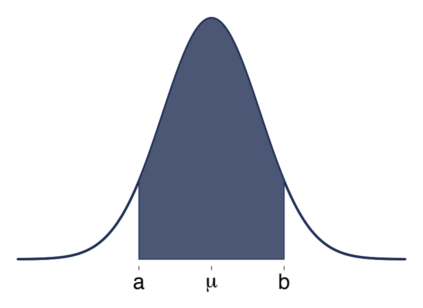


Probability density function properties

Properties of the probability density function (PDF), $f(x)$, of random variable X:

3. The probability that X lies between two values is the area under the density function graph between the two values:

$$P(a < X < b) = \int_a^b f(x)dx$$



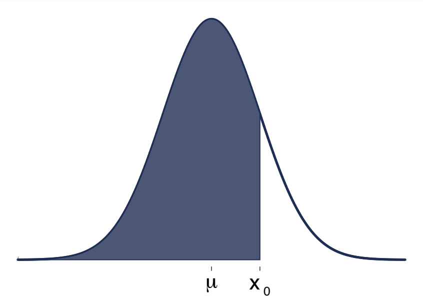
Cumulative density function (CDF)

Cumulative density function (CDF)

Definition

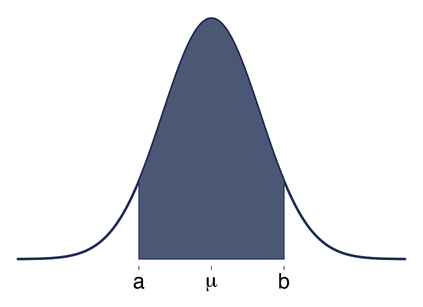
$F(x_o)$: The area under the probability density function $f(x)$ from the minimum x value up to x_0:

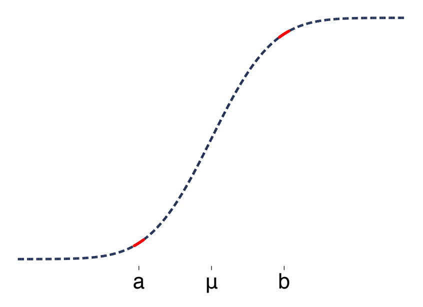
$$F(x_o) = \int_{x_m}^{x_0} f(x)dx$$



In some cases, $x_m = -\infty$.

Relationship between PDF & CDF





Prob. distributions

Key definitions: features of probability distributions

- Measures of central tendency: expected value
- Measures of variability: variance and standard deviation

Note: We refer to $E[Y]$ as the first moment of Y, $E[Y^2]$ as the second moment, $E[Y^3]$ as the third moment, etc.

Expected value discrete random variables

- The expected value of discrete random variable X:

$$E[X] = \mu = \sum_x xP(x)$$

- Long-run average value of the random variable X over many repeated trials
- Weighted average of possible outcomes, where weights are the probabilities of that outcome
- Also called the mean or expectation of X

Expected value of discrete random variables

Example 2

Recall an experiment in which we flip a coin 3 times. Let X be the number of heads.

X	0	1	2	3
P(x)	0.125	0.375	0.375	0.125

What is the expected value of X?

$$
\begin{aligned}
E[X] = \mu = \sum_x xP(x) \\
&= (0)P(X=0) + (1)P(X=1) + (2)P(X=2) + (3)P(X=3) \\
&= 0 + 0.375 + 0.75 + 0.375 \\
&= 1.50
\end{aligned}
$$

Variance/standard deviation

Variance of discrete random variable X **Definition**

$$\sigma^2 = E[(X-\mu)^2] = \sum_x (x-\mu)^2 P(x)$$

or

$$\sigma^2 = E[(X-\mu)^2] = \sum_x x^2 P(x) - \mu^2$$

Standard deviation of discrete random variable X **Definition**

$$\sigma = |\sqrt{\sigma^2}| = \sqrt{\sum_x (x-\mu)^2 P(x)|}$$

Linear functions of random variables

Let $W = a + bX$, where X has mean μ_X and variance σ_X^2, and a and b are constants:

- The mean of W is:

$$\mu_W = E[a + bX] = a + b\mu_X$$

- the variance of W is:

$$\sigma_W^2 = Var[a + bX] = b^2\sigma_X^2$$

- the standard deviation of W is:

$$\sigma_W = |b|\sigma_X$$

Joint distributions

Joint probability distributions

What about when we have two (or more) random variables?

Joint probability distribution **Definition**

Express the probability that $X = x$ and $Y = y$ simultaneously:

$P(x, y) = P(X = x \cap Y = y)$

Independence

Independence of X and Y **Definition**

$$X \text{ and } Y \text{ independent } \iff P(x, y) = P(x)P(y)$$

That is, joint probability distribution is the product of their marginal probability functions for all possible values. This can be extended to k random variables

Conditional probability distributions

Conditional probability distribution **Definition**

The conditional probability distribution of random variable Y expresses probability that $Y = y$ conditional on $X = x$:

$$P(y|x) = \frac{P(x, y)}{P(x)}$$

Similarly,

$$P(x|y) = \frac{P(x, y)}{P(y)}$$

Conditional probability distributions: example

Example 3

The probability that the air conditioning breaks at an old factory depends on whether it is a hot day or a cold day.

- $X = 1$ if air conditioning breaks, 0 otherwise
- $Y = 1$ if it is a hot day, 0 otherwise
- Suppose $P(0,0) = 0.4$, $P(0,1) = 0.2$, $P(1,0) = 0.1$, $P(1,1) = 0.3$
- *What is the conditional marginal probability distribution of X if it is a hot day?*

Conditional probability distributions: example

	Cool day ($Y = 0$)	Hot day ($Y = 1$)
AC works ($X = 0$)	0.4	0.2
AC breaks ($X = 1$)	0.1	0.3

$$P(X = 0|Y = 1) = \frac{P(0,1)}{P(1)} = \frac{0.2}{0.2 + 0.3} = 0.4$$

$$P(X = 1|Y = 1) = \frac{P(1,1)}{P(1)} = \frac{0.3}{0.2 + 0.3} = 0.6$$

Conditional expectation and variance **Definition**

We use conditional distributions to calculate the conditional expectation and conditional variance:

$$E[Y|X = x] = \sum_{i=1}^{k} y_i P(Y = y_i|X = x)$$

$$Var[Y|X = x] = \sum_{i=1}^{k} [y_i - E(Y|X = x)]^2 P(Y = y_i|X = x)$$

Covariance

- Let X and Y be discrete random variables with means μ_X and μ_Y
- The covariance between X and Y is the expected value of the product of their mean deviations

$$
\begin{aligned}
Cov(X, Y) &= E[(X - \mu_x)(Y - \mu_Y)] \\
&= \sum_x \sum_y (x - \mu_x)(y - \mu_y)P(x, y)
\end{aligned}
$$

Covariance and independence

- The covariance measures the direction of the **linear** relationship between two variables *(sometimes called "linear dependence")*.
- If two random variables X and Y are statistically independent, $\Rightarrow Cov(X, Y) = 0$.
- The converse is not necessarily true. $Cov(X, Y) = 0 \not\Rightarrow$ statistical independence.

Correlation

We can standardize the covariance between X and Y by dividing by their standard deviations to get the correlation between X and Y.

$$\rho = Corr(X, Y) = \frac{Cov(X, Y)}{\sigma_X \sigma_Y}$$

ρ is "unitless," $-1 \leq \rho \leq 1$

General rules: Linear sums and differences

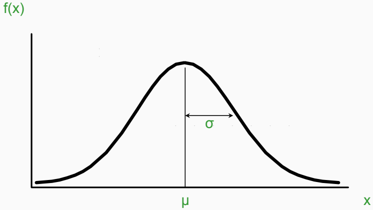
Handy relationships to remember:

$$
\begin{aligned}
E[aX + bY] &= a\mu_X + b\mu_Y \\
Var(aX + bY) &= a^2\sigma_X^2 + b^2\sigma_Y^2 + 2abCov(X, Y) \\
Var(aX - bY) &= a^2\sigma_X^2 + b^2\sigma_Y^2 - 2abCov(X, Y) \\
Cov(aX + b, cY + d) &= acCov(X, Y)
\end{aligned}
$$

Normal Dist.

Normal distribution

- Location determined by the mean, μ.
- Spread determined by standard deviation, σ.
- Bell-shaped & symmetrical
- Mean $=$ median $=$ mode
- Infinite range, $-\infty < x < \infty$



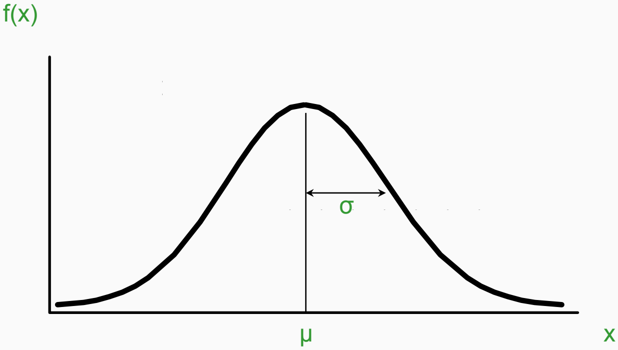
Normal distribution

- Distribution of sample means approach normal distribution with “large” sample size (Central Limit Theorem)
- Easy to compute probabilities!

A family of distributions

- Each distribution characterized entirely by μ and σ.
- We write the following for each distribution:

$$X \sim N(\mu, \sigma^2)$$



Normal PDF

Normal probability density function:

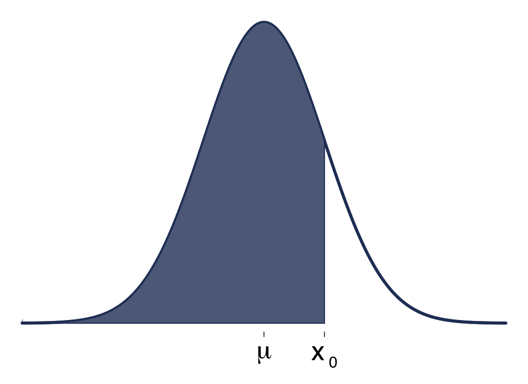
$$f(x) = \frac{1}{\sqrt{2\pi\sigma^2}} e^{-(x-\mu)^2/2\sigma^2}$$

This is difficult to work with directly! We will use probability tables.

Normal CDF

For $X \sim N(\mu, \sigma^2)$, the cumulative distribution function is:

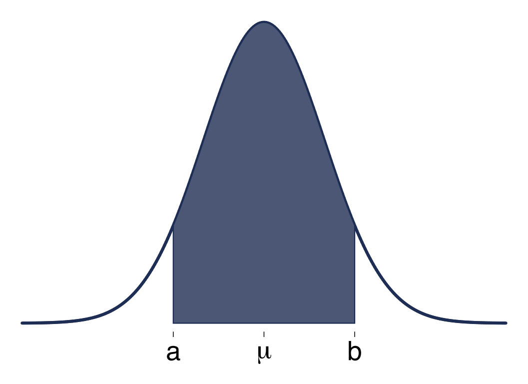
$$F(x_0) = P(X \leq x_0)$$



Finding normal probabilities

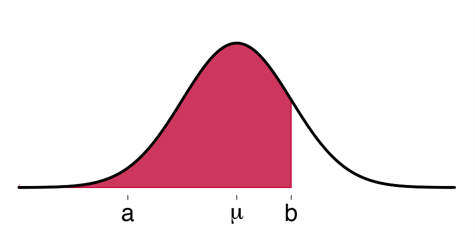
The probability for a range of values is measured by the area under the curve:

$$P(a < X < b) = F(b) - F(a)$$

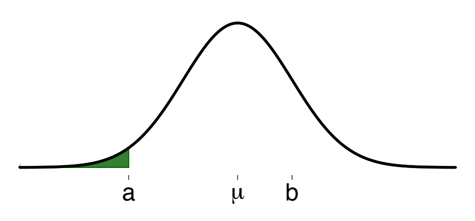


Finding normal probabilities

$$F(b) = P(X < b)$$



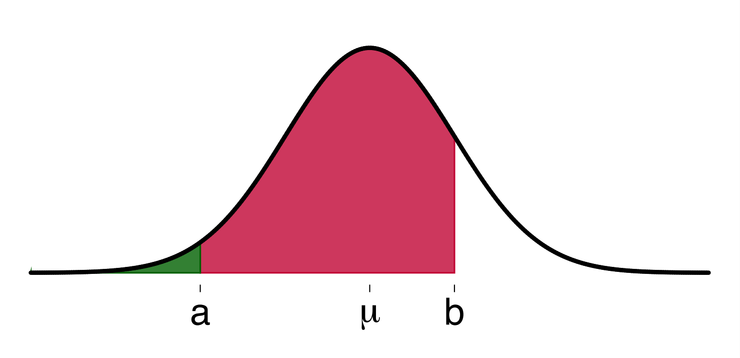
$$F(a) = P(X < a)$$



Finding normal probabilities

The probability for a range of values is measured by the area under the curve:

$$P(a < X < b) = F(b) - F(a)$$

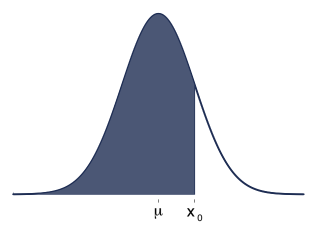


Finding normal probabilities

Things to note:

- $P(X \leq x_0) = P(X < x_0)$

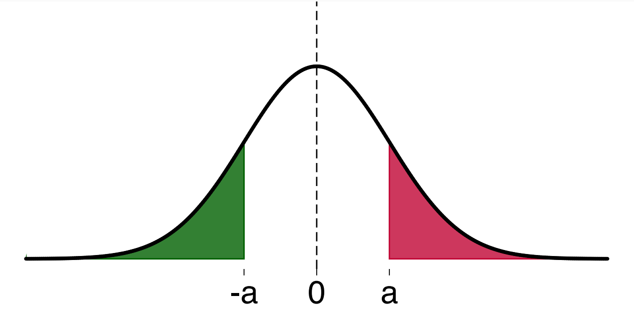
- $P(X < x_0) = 1 - P(X > x_0) \Rightarrow$
  $P(X > x_0) = 1 - P(X < x_0)$



Finding normal probabilities

Things to note:

- $P(X < -a) = P(X > a)$.



Recap: Linear functions of random variables

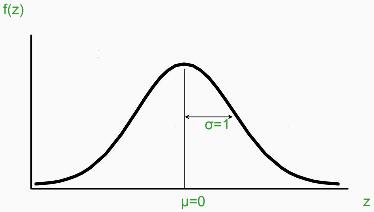
Special case: standardized random variable.

$$Z = \frac{X - \mu_X}{\sigma_X}$$

which has $\mu_Z = 0$ and $\sigma_Z^2 = 1$

The standard normal distribution

- *Any* normal distribution can be transformed into the standardized normal distribution ($Z \sim N(0, 1)$):



- We transform X units into Z units by subtracting the mean of X and dividing by its standard deviation:

$$Z = \frac{X - \mu_X}{\sigma_X}$$

Example: normal probabilities

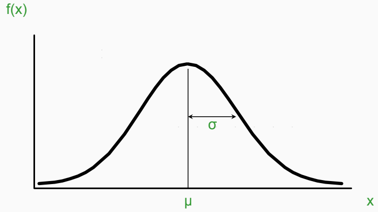
Example 4

If $X \sim (100, 50^2)$, what is the Z-value for $X = 200$?

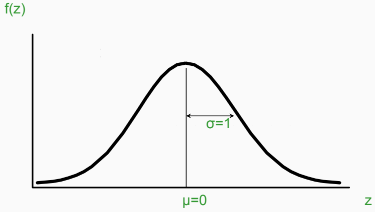
$$
\begin{aligned}
Z &= \frac{X - \mu_X}{\sigma_X} \\
&= \frac{200 - 100}{50} \\
&= 2.0
\end{aligned}
$$

Hence, $X = 200$ is **two standard deviations** (2 increments of 50 units) above the mean of 100.

Comparing X and Z units



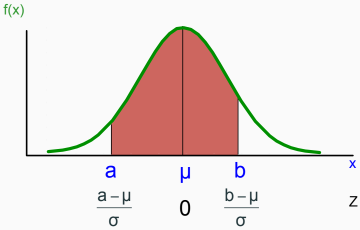
$\Leftrightarrow$



Note that the distribution is the same, only the scale has changed.

We can express the problem in original units (X) or standardized units (Z)

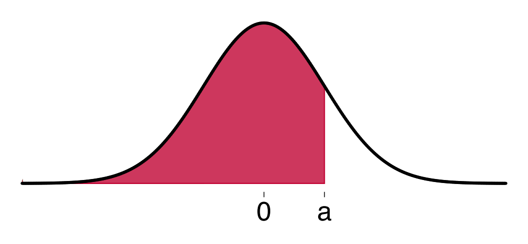
Finding normal probabilities



$$P(a < X < b) = P(\frac{a-\mu}{\sigma} < Z < \frac{b-\mu}{\sigma})$$
$$= F(\frac{b-\mu}{\sigma}) - F(\frac{a-\mu}{\sigma})$$

Standard normal distribution table

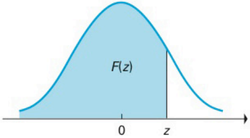
- The standard normal distribution table (available on Blackboard) shows values of the cumulative normal distribution function.
- For a given Z-value a, the table shows $F(a)$



Standard normal distribution table

Appendix Tables

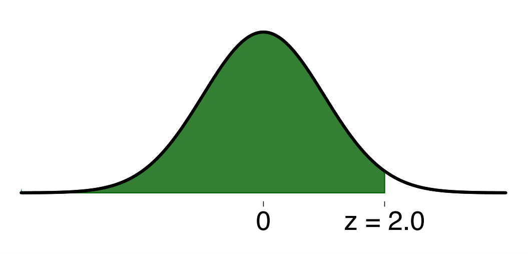
Table 1 Cumulative Distribution Function, $F(z)$, of the Standard Normal Distribution Table



z	0	0.01	0.02	0.03	0.04	0.05	0.06	0.07	0.08	0.09
0.0	0.5000	0.5040	0.5080	0.5120	0.5160	0.5199	0.5239	0.5279	0.5319	0.5359
0.1	0.5398	0.5438	0.5478	0.5517	0.5557	0.5596	0.5636	0.5675	0.5714	0.5753
0.2	0.5793	0.5832	0.5871	0.5910	0.5948	0.5987	0.6026	0.6064	0.6103	0.6141
0.3	0.6179	0.6217	0.6255	0.6293	0.6331	0.6368	0.6406	0.6443	0.6480	0.6517
0.4	0.6554	0.6591	0.6628	0.6664	0.6700	0.6736	0.6772	0.6808	0.6844	0.6879
0.5	0.6915	0.6950	0.6985	0.7019	0.7054	0.7088	0.7123	0.7157	0.7190	0.7224
0.6	0.7257	0.7291	0.7324	0.7357	0.7389	0.7422	0.7454	0.7486	0.7517	0.7549

Finding normal probabilities

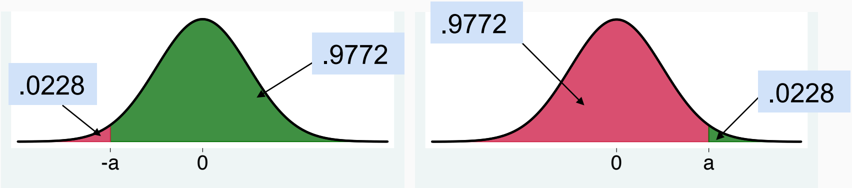
$$P(Z < 2.00) = 0.9772$$



Finding normal probabilities

For negative Z-values, recall that the distribution is symmetric:

$$P(Z < -a) = 1 - P(Z < a)$$



Type A: find probabilities, given $X \sim N(a, b)$

Example: A cupcake factory's daily production of cupcakes is normally distributed, with an average of 5,100 cupcakes per day and a standard deviation of 1,200 cupcakes. What is the probability that the factory produces more than 6,000 cupcakes tomorrow?

1. Draw normal curve for the problem in terms of X
2. Translate X-values to Z-values
3. Break into pieces of the form $F(Z < z)$
4. Use the cumulative normal table

Type B: find X-value, given probabilities

Example: A cupcake factory's daily production of cupcakes is normally distributed, with an average of 5,100 cupcakes per day and a standard deviation of 1,200 cupcakes. There is a 10% chance that the factory produces fewer than how many cupcakes tomorrow?

1. Find the Z-value for the known probability
2. Convert to X units using the formula:

$$X = \mu + Z\sigma$$

General rounding guidelines

Common z-values:

$F(z)$	0.90	0.95	0.975	0.99
z	1.282	1.645	1.960	2.326